ON THE STABILIZATION OF STEADY-STATE MOTIONS OF A NONLINEAR CONTROL SYSTEM IN THE CRITICAL CASE OF A PAIR OF PURE IMAGINARY ROOTS

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The problem of stabilization of steady-state motions of a nonlinear control system in the critical case of a pair of pure imaginary roots, is considered in this paper. Nonanalytical control devices and Liapunov functions of a special kind are used. Methods of solution of the problem based on the theory of the stability of motion [1 to 4] and on methods developed in [5 to 7] are considered. An example is studied.

1. Statement of the problem and auxiliary transformation. Let us consider the equations of the perturbed motion of a controlled object

$$dx/dt = Ax + Bu + g(x, u) \qquad (x \in \{R^n\},\ u \in \{R^m\}) \tag{1.1}$$

where x is the n-dimensional vector of the perturbation; u is the m-dimensional vector of the control; $g(x, y)$ are the terms in x and u of order higher than the first; A , B are constant matrices of corresponding orders. We shall assume that the control u is not affected by disturbances and does not consist of small terms of order higher than the order of x . We shall assume firthermore that all the corfficients of Equations (1.1) are real, and that $g(x, u)$ is analytical with respect to x and u .

Let us assume that for $u \equiv 0$ the unperturbed motion $x = 0$ of the system (1.1) is not asymptotically stable. It is necessary to design a stabilizing control (controller) $u = u(x)$ such, that the unperturbed motion $x = 0$ of the system (1.1) with this control becomes asymptotically stable in the sense of Liapunov.

Let us assume we have the critical case of a pair of pure imaginary roots [7]. Then, as shown in [3 and 7], the system (1.1) can be brought into the following form by a nondegenerated transformation of the variables:

$$\frac{d\xi}{dt} = -\lambda\eta + X(\xi, \eta, v, u), \qquad \frac{d\eta}{dt} = \lambda\xi + Y(\xi, \eta, v, u)$$
$$\frac{dv}{dt} = A_0 v + B_0 u + a\xi + b\eta + Z(\xi, \eta, v, u) \tag{1.2}$$

Here ξ and η are scalars, v is a $(n-2)$-dimensional vector, A_0 is a $(n-2)\times(n-2)$-matrix, B_0 is a $(n-2)\times m$-matrix, a and b are $(n-2)$-vectors, X, Y, Z are the terms in ξ, η, v, u, of order higher than the first and the control $u(v, \xi, \eta)$ is supposed to be nonanalytical.

2. Choice of the controller. It can be checked that the system

$$\frac{dv}{dt} = A_0 v + B_0 u \qquad (v \in \{R^{n-2}\},\ u \in \{R^m\}) \tag{2.1}$$

satisfies the condition of stabilizability [7] and that, consequently, one can design for it a controller of the form

$$u_0(v) = Pv \tag{2.2}$$

where P is some $m \times (n-2)$-matrix.

For brevity let us introduce the following notation:

$$x_* = \operatorname{sign} x = \begin{cases} 1 & \text{for } x \geqslant 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{2.3}$$

We shall seek for the system (1.2) a controller of the form

$$u^j(v, \xi, \eta) = u_0^j(v) + \sum_{p,\, q=0}^{1} \sum_{s+k=1}^{\infty} \alpha_{skpq}^{\ j} \xi^s \eta^k \xi_*^{\ p} \eta_*^{\ q} \tag{2.4}$$

From here on, we shall assume $s \geqslant 0$ and $k \geqslant 0$.

If the coefficients in the series (2.4) are constrained by the condition

$$\alpha_{0k1q}^{\ j} = \alpha_{s0p1}^{\ j} = 0 \qquad (j = 1, \ldots, m) \tag{2.5}$$

we obtain a continuous controller. For $p = q = 0$, this yields an analytical controller.

We shall bring the control (2.4) into the system (1.2) and we shall try to transform it in such a manner that the problem of the stability of the entire system could be solved by considering some simplified system of the second order, corresponding to a pair of pure imaginary roots. We shall choose the undetermined coefficients of the control (2.4) on the basis of the criteria of stability of the simplified system, the construction of which we shall consider now.

In agreement with the method of Liapunov [1 to 3] we shall consider the system of partial differential equations

$$\frac{\partial z}{\partial \xi}[-\lambda\eta + X(\xi, \eta, z, u)] + \frac{\partial z}{\partial \eta}[\lambda\xi + Y(\xi, \eta, z, u)] =$$
$$= A_0 z + B_0 u + a\xi + b\eta + Z(\xi, \eta, z, u) \tag{2.6}$$

Here, z is a $(n-2)$-vector, and $u = u(z, \xi, \eta)$ in agreement with (2.2), (2.4). According to Liapunov's theorem [1 and 3], there exists only one solution for the system (2.6) in the analytical case when $p = q = 0$.

Using the nonanalytical control (2.4), we shall search for a solution of the system (2.6) in the form of series

$$z_i = \sum_{p,\,q=0}^{1} \sum_{s+k=1}^{\infty} c_{skpq}^{\,i} \xi^s \eta^k \xi_*^{\,p} \eta_*^{\,q} \qquad (i = 1, \ldots, n-2) \tag{2.7}$$

with the undetermined coefficients c_{skpq}^{i}, choosing the latter such that Equation (2.6) is satisfied identically in ξ, η, ξ_*, η_*.

We shall substitute (2.7) into (2.6) and gather the components which include factors of ξ_*, η_*. Then Equation (2.6) is broken into four relations corresponding to the combinations of the indices

$$(p, q) = (0, 0),\ (0, 1),\ (1, 0),\ (1, 1) \tag{2.8}$$

whereupon the coefficients c_{skpq}^{i} which should be determined, are distributed in these four relations in such a manner that they can be determined in succession.

Thus, in the general case there is only one solution of the system (2.6) which can be represented over the complete range of variation of the variables by the single relation (2.7). We shall point out, that on the basis of the definition of the function $x_* = \operatorname{sign} x$ (2.3), the vector-function $z(\xi, \eta)$ is differentiable any number of times everywhere, except on the surfaces $\xi = 0$ and $\eta = 0$.

Substituting the control $u(v, \xi, \eta)$ (2.4) into Equation (1.2) and replacing the vector v by the vector z (2.7) in the obtained relation, we get the second order system

$$\begin{aligned} \frac{d\xi}{dt} &= -\lambda\eta + \sum_{p,\,q=0}^{1} \sum_{s+k=2}^{\infty} a_{skpq} \xi^s \eta^k \xi_*^{\,p} \eta_*^{\,q} \\ \frac{d\eta}{dt} &= \lambda\xi + \sum_{p,\,q=0}^{1} \sum_{s+k=2}^{\infty} b_{skpq} \xi^s \eta^k \xi_*^{\,p} \eta_*^{\,q} \end{aligned} \tag{2.9}$$

The system (2.9) is obtained from the system (1.2) by means of the Liapunov transformation

$$v_i = w_i + z_i \qquad (i = 1, \ldots, n-2) \tag{2.10}$$

leaving aside all equations corresponding to noncritical roots and writing $w_i \equiv 0$ in the remaining. In the sequel, we shall confine ourselves to those controls (2.4) for which the transformation (2.10) is continuous. Discontinuities may occur on the surfaces $\xi = 0$, $\eta = 0$ in the right-hand sides of the transformed system. However we can verify the validity of the principle of reduction [3] (pp.373-382) also in the above case, so that the solution of the stability problem of the system (1.2) can be found from a study of the system (2.9).

N o t e 2.1 . Let us formulate the principle of reduction for this problem. Substituting into (1.2) the control $u(v, \xi, \eta)$ in agreement with (2.4), we get the system

$$\frac{d\xi}{dt} = -\lambda\eta + X_u(\xi, \eta, v), \qquad \frac{d\eta}{dt} = \lambda\xi + Y_u(\xi, \eta, v) \tag{2.11}$$

$$\frac{dv}{dt} = (A_0 + B_0 P)\, v + Z_u(\xi, \eta, v) \tag{2.12}$$

Let us leave Equations (2.12) aside and let $v \equiv 0$ in Equations (2.11). Then, we get the simplified system of second order

$$\frac{d\xi}{dt} = -\lambda\eta + X_u(\xi, \eta, 0) \qquad \frac{d\eta}{dt} = \lambda\xi + Y_u(\xi, \eta, 0) \tag{2.13}$$

Let us assume that in the expansions of $X_u(\xi, \eta, v), Y_u(\xi, \eta, v)$ q is the lowest order of the terms which are functions of v_i $(i = 1, \ldots, n-2)$ and that the lowest order with respect to v_i of these terms is $r \leqslant q$.

Theorem. We shall assume that the stability, asymptotic stability or instability of the unperturbed motion $\xi = \eta = 0$ of the system (2.13) is independent of the terms of order higher than N. Then, if the expansion of the vector $Z_u(\xi, \eta, 0)$ begins with terms of order not lower than p, where

$$p \geqslant \frac{N + 1 - q + 2r}{r} \tag{2.14}$$

then the unperturbed motion $\xi = \eta = v_i = 0$ for the total system (2.11), (2.12) is correspondingly stable, asymptotically stable or unstable.

The proof of this theorem is basically the same as that given for analytical systems in Malkin's book [3].

If the solution (2.7) of the partial differential equation (2.6) is continuous in the terms of order $s + k \leqslant p - 1$, then the condition (2.14) can be replaced by a weaker condition [3] (p.386) for which

$$p' \geqslant \frac{N + 1 - q + r}{r} \tag{2.15}$$

In order to transform the system (2.11), (2.12) into a system satisfying the conditions of the theorem, and by the same token construction the simplified system (2.9), it is necessary to compute the continuous functions z_i (2.7) up to their terms of order $p - 2$ and check the existence of the continuous solution of (2.6) in the terms of order $p - 1$. If such a solution does not exist, then one must try to eliminate the terms of order $p - 1$ in the expansion of the vector $Z_u(\xi, \eta, 0)$ by a proper choice of the controls.

Note 2.2. It is sometimes expedient to utilize the continuous controller

$$u^j = u_0^j(v) + \sum_{l=-\infty}^{1} \sum_{s+k+l=1}^{\infty} \alpha_{skl}^j \xi^s \eta^k \zeta^l \qquad (j = 1, \ldots, m) \tag{2.16}$$

which contains the nonanalytical function (*)

$$\zeta = \sqrt{\beta\xi^2 + \gamma\eta^2} \qquad (\beta, \gamma > 0) \tag{2.17}$$

Then the Liapunov transformation (2.10) is continuous, and this eliminates the complications, connected with the possible appearance of discontinuities in (2.10) when the controller (2.4) is used.

3. Choice of the Liapunov functions. In the investigation of the stability of motion of the system (2.9) with a nonanalytical right-hand side it is useful to consider nonanalytical Liapunov functions. The condition

$$\varphi_k(\nu x, \mu y) = |\nu|^k \varphi_k(x, y) \qquad (|\nu| = |\mu| \neq 0;\ k = 1, 2, \ldots) \tag{3.1}$$

separates those functions for which checking the sign definiteness does not become more complicated than by using the methods known for analytical functions. The functions (3.1) are even, continuous and monotonous along any path starting from the origin of the coordinates; they satisfy the condition $\varphi_k(0, 0) = 0$, keep their sign in all the points of the straight lines

*) This was pointed out to the author by N.N. Krasovskii.

$$x \pm \alpha y = 0 \qquad (\alpha \text{ is arbitrary}) \tag{3.2}$$

and are quantities of the kth order with respect to the first order variables x and y.

The functions $\varphi_k(x, y)$ are sign definite if and only if Equations

$$\varphi_k(\rho, 1) = 0, \qquad \varphi_k(1, \rho) = 0 \tag{3.3}$$

do not have any roots $\rho \geqslant 0$. This could be proved on the basis of the properties of the functions (3.1) pointed out above.

It is expedient to construct a Liapunov function in the form (2.7). The terivative dV/dt will also have the same form. When the sign-definiteness must be verified, we shall take those functions of the class (2.7) which satisfy the condition (3.1). All such functions can be expressed in the form

$$\psi_k(x, y) = \sum_{i+j=k} \gamma_{ij}^k (xx_*)^i (yy_*)^j \qquad (k = 1, 2, \ldots) \tag{3.4}$$

4. First method of solution. We shall assume that the construction is done in such a manner (see the example) that the Liapunov transformation (2.10) is continuous. If at some stage any choice of a nonanalytical control from (2.4) leads to discontinuous transformations (2.10), then one must take analytical controls at that stage and the previous ones (when necessary), (see example). We write Equations (2.9) in the form

$$\begin{aligned} d\xi/dt &= -\lambda\eta + X_2(\xi, \eta) + X_3(\xi, \eta) + \ldots \\ d\eta/dt &= \lambda\xi + Y_2(\xi, \eta) + Y_3(\xi, \eta) + \ldots \end{aligned} \tag{4.1}$$

where X_k, Y_k represents the ensemble of the terms of kth order in Equations (2.9). We try to construct a Liapunov function satisfying the conditions of the asymptotic stability theorem in the form

$$V = \xi^2 + \eta^2 + V_3(\xi, \eta) + V_4(\xi, \eta) + \ldots \tag{4.2}$$

where $V_k(\xi, \eta)$ is a continuous function of the kth order of the class (2.7)

$$V_k(\xi, \eta) = \sum_{p, q=0}^{1} \sum_{i+j=k} \beta_{ijpq}^{\,k} \xi^i \eta^j \xi_*^{\,p} \eta_*^{\,q} \qquad (k \geqslant 3) \tag{4.3}$$

$$\beta_{0j1q}^{\,k} = \beta_{i0p1}^{\,k} = 0 \qquad (k = 3, 4, \ldots) \tag{4.4}$$

The total derivative dV/dt, on the basis of Equations (4.1), is obtained in the form

$$\frac{dV}{dt} = f_3(\xi, \eta) + f_4(\xi, \eta) + \cdots \tag{4.5}$$

$$f_k(\xi, \eta) = \lambda \left(\xi \frac{\partial V_k}{\partial \eta} - \eta \frac{\partial V_k}{\partial \xi} \right) + F_k(\xi, \eta) \tag{4.6}$$

$$F_k(\xi, \eta) = 2\xi X_{k-1} + 2\eta Y_{k-1} + \sum_{i+j=k+1} \left(\frac{\partial V_i}{\partial \xi} X_j + \frac{\partial V_i}{\partial \eta} Y_j \right) \tag{4.7}$$

$$(k \geqslant 3; \quad i \geqslant 3; \quad X_0 = Y_0 = X_1 = Y_1 = 0)$$

The function $F_k(\xi, \eta)$ depends on the functions $V_3, \ldots, V_{k-1}$ and can be computed from (4.7), if these functions are known. Let us consider the ensemble of the terms of third order in (4.5)

$$f_3(\xi, \eta) = \lambda \left(\xi \frac{\partial V_3}{\partial \eta} - \eta \frac{\partial V_3}{\partial \xi} \right) + 2\xi X_2 + 2\eta Y_2 \tag{4.8}$$

We shall require that for these terms the condition

$$f_3(\xi, \eta) = -\psi_3(\xi, \eta) \tag{4.9}$$

be fulfilled, where $\psi_3(\xi, \eta)$ is positive-definite function from (3.4). For instance one can take

$$\psi_3(\xi, \eta) = \gamma_{30}{}^3\xi^3\xi_* + \gamma_{03}{}^3\eta^3\eta_* \quad (\gamma_{ij}{}^3 > 0) \tag{4.10}$$

The condition (4.9) becomes a partial differential equation with respect to the function $V_3(\xi, \eta)$ which can be solved by the method of undetermined coefficients. We shall substitute in (4.9) Expression $V_3(\xi, \eta)$ in the form (4.3) and we shall not fix beforehand the coefficients $\gamma_{ij}{}^3$ of the function $\psi_3(\xi, \eta)$.

Then, we shall obtain a system of relations (A) consisting of: (1) the conditions of continuity (4.4), (2) the conditions of positive definiteness of $\psi_3(\xi, \eta)$ in the form of inequalities imposed on $\gamma_{ij}{}^3$, (3) linear equations with constant coefficents expressing the condition (4.9).

The stabilization is guaranteed by the control (2.4) if the coefficents α^j_{skpq}, β^3_{ijpq}, γ^3_{ij} can be chosen such that the system of relations (A) is satisfied. Thus, it is sufficient to consider only the terms of first order in (2.4) whereupon we get a family of controllers depending on the parameters $\gamma_{ij}{}^3$. If the relations (A) cannot be fulfilled because of the coupling, or the structure of the system (2.9), then the conditions on $\gamma_{ij}{}^3$ must be loosened, requiring only that the function $\psi^3(\xi, \eta)$ be positive semi-definite $\psi_3(\xi, \eta) \geqslant 0$. This system of relations (A′) can always be satisfied. It is sufficient, for instance, to equate to zero all the coefficents α^j_{skpq} entering $f_3(\xi, \eta)$ in (4.8), to take $\psi_3 \equiv 0$ and to find a sole analytical solution for $V_3(\xi, \eta)$. Then it is necessary to consider in (4.5) the set of terms corresponding to the next measurement.

We shall note that for any function $V_k(\xi, \eta)$ of the form (4.3) for $\xi = \cos\theta$, $\eta = \sin\theta$, the equality

$$\frac{dV_k}{d\theta} = \left(\xi\frac{\partial V_k}{\partial\eta} - \eta\frac{\partial V_k}{\partial\xi}\right)\Bigg|_{\substack{\xi=\cos\theta\\ \eta=\sin\theta}} \tag{4.11}$$

is valid everywhere except on the surfaces $\xi = 0$, $\eta = 0$.

We shall denote by M_3 the set consisting of the points of the phase plane for which $\psi_3(\xi, \eta) = 0$ (excluding the origin $\xi = \eta = 0$) and some neighborhood of these points. If $\psi_3 \equiv 0$, M_3 consists of the whole surface except the origin, if $\psi_3(\xi, \eta) \geqslant 0$, M_3 consists of the open domains limited by the paths (3.2) infinitely close to these straight lines (3.2) for which $\psi_3 = 0$. Let us consider the set of the fourth order terms in (4.5)

$$f_4(\xi, \eta) = \lambda\left(\xi\frac{\partial V_4}{\partial\eta} - \eta\frac{\partial V_4}{\partial\xi}\right) + F_4(\xi, \eta) \tag{4.12}$$

We require that for those terms the condition

$$f_4(\xi, \eta) = -\psi_4(\xi, \eta) + c_4(\xi^2 + \eta^2)^2 \qquad (c_4 = \text{const}) \tag{4.13}$$

be satisfied for the set M_3. Here $\psi_4(\xi, \eta)$ is some function from (3.4), the coefficents γ^4_{ij} of which we shall not fix. We shall calculate the integral

$$c_4 = c_4(\gamma_{ij}^4) = \frac{1}{2\pi}\int_0^{2\pi} [F_4(\cos\theta, \sin\theta) + \psi_4(\cos\theta, \sin\theta)]\, d\theta \qquad (4.14)$$

The imtegral (4.14) determines the linear relation between c_4 and the parameters γ_{ij}^4 for which it is possible to find a solution for the system of linear equations obtained when the partial differential equation (4.13) is solved by the method of undetermined coefficients. Let us consider the function

$$\Phi_4 = \psi_4(\xi, \eta) - c_4(\xi^2 + \eta^2)^2 \qquad (4.15)$$

which belongs to the class (3.4). The system of relations (B) and (B′) are obtained from relations (A) and (A′) when the functions $\psi_3(\xi, \eta)$ are replaced by the function $\Phi_4(\xi, \eta)$ of (4.15) and the linear equations necessary for (4.9) by the linear equations determined for (4.13). Thus, the condition of sign definiteness or sign invariability of Φ_4 have to be established only for the set M_3. Since $\Phi_4(\xi, \eta)$ is chosen in the class (3.4), its condition of positiveness on the set M_3 requires, according to (3.3) that the roots $\rho \geqslant 0$ of Equation $\psi_3(\rho, 1) = 0$ yield positive values to the polynomial

$$\Phi_4(\rho, 1) = \gamma_{04}^4 + \gamma_{13}^4\rho + \gamma_{22}^4\rho^2 + \gamma_{31}^4\rho^3 + \gamma_{40}^4\rho^4$$

and simultaneously, that the roots $\rho \geqslant 0$ of Equation $\psi_3(1, \rho) = 0$ yield positive values to the polynomial

$$\Phi_4(1, \rho) = \gamma_{04}^4\rho^4 + \gamma_{13}^4\rho^3 + \gamma_{22}^4\rho^2 + \gamma_{31}^4\rho + \gamma_{40}^4$$

The coefficients β_{ijpq}^4, γ_{ij}^4 and the remaining coefficients α_{skpq}^j must be chosen such that the relations (B) or at least the relations (B′) be satisfied. In the latter case it is necessary to determine the set M_4 consisting of the neighborhood of the points M_3 for which $\Phi_4(\xi, \eta) = 0$, and consider the ensemble of the terms of the following measurement and so on.

With this procedure, we obtain: (i) relations (A), (A′) for the odd orders of (4.5), (ii) relations (B), (B′) for the even orders $k = 2n$ with functions in the right-hand side of (4.13) of the form

$$\Phi_k(\xi, \eta) = \psi_k(\xi, \eta) - c_k(\xi^2 + \eta^2)^{\frac{1}{2}k} \qquad (4.16)$$

and (iii) the system of enclosed sets

$$\{R^2\} = M_2 \supset M_3 \supseteq M_4 \supseteq \ldots \qquad (4.17)$$

The conditions of sign definiteness (A), (B), or of sign invariability (A′), (B′) for the terms of the kth order in (4.5) are considered on the set M_{k-1}.

Let us assume that the set M_{i-1} (i is an even number) includes the whole phase plane except the origin of the coordinates; this means that for $3 \leqslant k$, $s \leqslant i - 1$ the identities $\psi_k(\xi, \eta) = \Phi_s(\xi, \eta) \equiv 0$ are satisfied. We shall determine a system of relations (C) by replacing in the relations (B) the conditions of positiveness on the set M_{i-1} by conditions of negativeness.

The controller will be designed, if an instant comes when after some step $l \geqslant 3$ the set M_l is empty, i.e. the relations (A) and (B) are satisfied.

Whereupon for the terms of lower order of $f_k(\xi, \eta)$ in (4.5), $3 \leqslant k \leqslant l-1$ the relations (A′) or (B′) are fulfilled, and in addition the right-hand sides of the noncritical equations of the transformed system for $w_1 = 0$ do not contain terms of order

$$s \leqslant \frac{l-q+r}{r}$$

(see note 2.1 and Formula (2.14) for $N = l - 1$) .

If, for the same conditions for the right-hand sides of the noncritical equations, for any possible values of $\alpha^j_{skpq}, \beta^m_{ijpq}, \gamma^m_{ij}$ $(m \leqslant l)$ the relations (C) are satisfied on the basis of the coupling or the structure of the system (4.1), for some $l \geqslant 4$ and when the conditions

$$\psi_k(\xi, \eta) = \Phi_s(\xi, \eta) \equiv 0$$

for some $3 \leqslant k, s \leqslant l-1$ are fulfilled, then the stabilization by the control (2.4) is not possible

We shall note that once a first sign invariable function $\psi_k(\xi, \eta) \geqslant 0$ or $\Phi_k(\xi, \eta) \geqslant 0$ has been chosen, we can take the functions ψ_s, Φ_s $(s \geqslant k+1)$, which are sign changing in the phase plane and take nonnegative values on the sets M_{s-1} $(s \geqslant k+1)$.

The validity of the given statements follows from Liapunov's asymptotic stability theorem [1], Chetaev's instability theorem [2] and the principle of reduction for a quasi-analytical system.

N o t e 4.1 . Let us consider a particular case. Let $k \geqslant 3$ be the first index in (4.5) for which the relations (A′) or (B′) are fulfilled, whereupon the function $\psi(\xi, \eta)$ or $\Phi_k(\xi, \eta)$ is not identically equal to zero.

We shall assume that if we take $V_i(\xi, \eta) \equiv 0$ in (4.2) for $i > k$ then, according to (4.7), there follows $F_{k+1}(\xi, \eta) \equiv 0$. We shall denote the mentioned function by $\varphi_k(\xi, \eta)$ and we shall consider the functions

$$W = V + a\xi\eta^k - b\xi^k\eta \qquad (k \text{ odd}) \tag{4.18}$$

$$W = V + a\xi\eta^k\eta_* - b\xi^k\eta\xi_* \qquad (k \text{ even}) \tag{4.19}$$

where V is the function (4.2) constructed up to the terms of order k . The total derivative of the functions (4.18), (4.19) on the basis of Equations (2.9) can be written in the form

$$\frac{dW}{dt} = -\varphi_k(\xi, \eta) - \lambda a|\eta|^{k+1} + \lambda kb|\xi|^{k-1}\eta^2 + \lambda ka\xi^2|\eta|^{k-1} - \lambda b|\xi|^{k+1} + \ldots \tag{4.20}$$

where $|\xi| = \xi\xi_*$ is the absolute value of ξ . Whereupon, on the basis of the assumptions $V_i(\xi, \eta) \equiv 0$ for $i > k$, $F_{k+1} \equiv 0$, and according to (4.6) we shall have $f_{k+1}(\xi, \eta) \equiv 0$, consequently, the derivative dW/dt in (4.20) does not have any terms of order $k+1$ except for those written out. Since $\varphi_k(\xi, \eta) \geqslant 0$ is a function of the class (3.1), it can become equal to zero only on the straight lines (3.2). Let us suppose that $\xi = \alpha_i\eta$ are those straight lines. We shall choose the numbers a, b in (4.18) or (4.19) such that the inequality

$$a - kb|\alpha_i|^{k-1} - ka\alpha_i^2 + b|\alpha_i|^{k+1} > 0 \tag{4.21}$$

be satisfied.

Then, the set of terms of order $(k+1)$ of (4.20) is negative in the neighborhood of the straight lines $\xi = \alpha_i\eta$. If it is possible to choose a and b such that they satisfy simultaneously all the inequalities (4.21), then it is not necessary to consider the terms of higher order and the designed control stabilizes the system. In particular for a straight line $\xi = \alpha\eta$ such a choice of a, b is possible for any $0 \leqslant |\alpha| \leqslant \infty$.

5. Second method. The controller (2.4) can be designed by using the method of investigation of [3] in connection with an estimate of the sign of the contour integrals. Thus we obtain

$$dr \,/\, d\theta = c^2 F_2(\theta) + c^3 F_3(\theta) + \ldots, \quad r(0, c) = c, \quad c = \text{const} \tag{5.1}$$

The functions $F_l(\theta)$ ($l \geqslant 2$) are determined in sequence and depend on the coefficients $\alpha_{i\,kpq}^{j}$ in (2.4). Let us formulate the result.

The stabilization is guaranteed by the control (2.4) if the transformation (2.10) is continuous and the coefficients $\alpha_{i\,kpq}^{j}$ are chosen such that after some step $l \geqslant 2$ the conditions

$$J_l = \int_0^{2\pi} F_l(\theta)\, d\theta < 0, \qquad J_k = \int_0^{2\pi} F_k(\theta)\, d\theta = 0 \qquad (2 \leqslant k \leqslant l-1) \tag{5.2}$$

are satisfied and that, furthermore, the right-hand sides of the noncritical equations of the transformed system for $w_1 = 0$ do not contain terms of order $s \leqslant (l+1-q+r)\,/\,r$ (see (2.14) for $N = l$).

If, for any possible $\alpha_{i\,kpq}^{j}$, on the basis of the coupling or the structure of the system (2.9), the condition $J_l > 0$ is satisfied for some $l \geqslant 2$ and $J_k = 0$ for $2 \leqslant k \leqslant l-1$ then the stabilization by a control of the form (2.4) is not possible.

Note 5.1. This method can also be used to check for asymptotic stability when a control has been preliminary designed by means of a Liapunov function having a negative definite derivative. For that purpose the integrals (5.2) must be evaluated. Let $p \geqslant 3$ be the index of (4.5) for which the relations (A′) or (B′) are fulfilled. The unperturbed motion $\xi = \eta = 0$ will be asymptotically stable independently of the terms of order higher than $p-1$ if, and only if the conditions (5.2) are fulfilled for some $2 \leqslant l \leqslant p-1$.

6. Example (*). Let us consider a pendulum in gimbals, having two degrees of freedom and bearing a material sphere which can rotate around the arm of the pendulum. It is required to stabilize asymptotically the equilibrium position of the pendulum, using the moment u_1 which rotates the sphere around the arm and the moment u_2 in one of the planes of oscillation (see Fig.1). Let

$$\frac{d\theta}{dt} = x_1, \quad \theta = x_2, \quad \frac{d\varphi}{dt} = x_3, \quad \varphi = x_4, \quad \frac{d\psi}{dt} = \xi, \quad \psi = \eta \tag{6.1}$$

The equations of motion of the pendulum expressed with the accuracy up to the 4th order terms in the normal form of Cauchy have the form

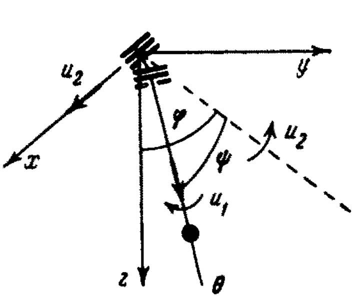

Fig. 1

$$\frac{dx_1}{dt} = u_1 + X_1^{(2)} + X_1^{(3)}, \qquad \frac{dx_2}{dt} = x_1 \tag{6.2}$$

$$\frac{dx_3}{dt} = -x_4 + u_2 + 2kx_1\xi + X_3^{(3)}, \qquad \frac{dx_4}{dt} = x_3 \tag{6.3}$$

$$\frac{d\xi}{dt} = -\eta - 2kx_1x_3 + Y^{(3)}, \qquad \frac{d\eta}{dt} = \xi \tag{6.4}$$

Here u_1, u_2 are controls proportional to the moments, $k > 0$ is a parameter, $X_i^{(s)}, Y^{(s)}$ ($s = 2, 3$) are ter of order $s > 1$, in respect to the variables x_i ($i = 1,\ldots,4$), u_j ($j = 1, 2$); ξ and η are quantities of the first oeder.

The variables x_i ($i = 1,\ldots, 4$) of Equations (6.2), (6.3) are fully controllable, while the variables ξ, η in the first approximation are not controllable and Equation (6.4) has a pair of pure imaginary roots $\lambda_{1,2} = \pm i$. The linear controller of the first

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*) This system was suggested for consideration by Krasovskii.

approximation control subsystem can be taken in the form $u_1^0 = -2x_1 - x_2$, $u_2^0 = -2x_3$.

1. We search for the complete system a controller (2.4) of the form

$$u_j = u_j^0 + u_j^{(1)} + u_j^{(2)} + u_j^{(3)} + \ldots \qquad (j = 1, 2) \tag{6.5}$$

where $u_j^{(s)}$ is the function of the sth order of (2.7).

We shall find at the beginning the first order terms of (6.5). We shall write the function of the first order from the relation (2.7)

$$z_i^{(1)} = p_i\xi + q_i\eta \qquad (i = 1, \ldots, 4) \tag{6.6}$$

where

$$p_i = p_0^i + p_1^i\xi_* + p_2^i\eta_* + p_3^i\xi_*\eta_* \tag{6.7}$$

and analogously for q_i ($i = 1, \ldots, 4$). From the requirement of continuity of the Liapunov transformation (2.10) and Equation (6.6) we obtain

$$p_j = p_0^i + p_1^i\xi_*, \qquad q_i = q_0^i + q_2^i\eta_* \tag{6.8}$$

We shall substitute (6.6) in Equation (2.6), established for the system (6.2), (6.3), (6.4), and express the terms of first order.

The necessary conditions of solvability of the partial differential equation (2.6) with respect to $z_i^{(1)}$ have the form

$$q_2 = p_1, \quad -p_2 = q_1, \quad q_4 = p_3; \quad -p_4 = q_3 \tag{6.9}$$

On the basis of (6.8), (6.9) the quantities p_i, q_i must be independent from ξ_*, η_*. Thus, the functions $u_j^{(1)}$ will be similarly independent from ξ_*, η_*, i.e. the first order terms of (6.5) are linear, and one can try to determine them, by means of procedure used to investigate the stability in analytical case [1 to 3, 8 and 9].

We set $u_j^{(1)} \equiv 0$, and we shall search for the terms of second order in (6.5). We can also take $p_i = q_i = 0$, i.e. $z_i^{(1)} \equiv 0$. The second order function from (2.7) is

$$z_i^{(2)} = a_i\xi^2 + b_i\xi\eta + c_i\eta^2 \qquad (i = 1, \ldots, 4) \tag{6.10}$$

where a_i, b_i and c_i, are functions of ξ_*, η_* of the same form as p_i (6.7). The conditions of continuity of $z_i^{(2)}$ have the form

$$a_i = a_0^i + a_1^i\xi_*, \qquad c_i = c_0^i + c_2^i\eta_* \tag{6.11}$$

We shall substitute (6.10) into Equation (2.6), set for the system (6.2), (6.3), (6.4) and we shall express the second order terms. The conditions of solvability of the partial differential equation (2.6) with respect to $z_i^{(2)}$ are obtained in the form

$$b_2 = a_1 = -c_1, \quad 2(c_2 - a_2) = b_1, \quad b_4 = a_3 = -c_3, \quad 2(c_4 - a_4) = b_3 \tag{6.12}$$

From (6.11) and (6.12) we obtain the equalities

$$b_2 = a_1 = -c_1 = \nu, \qquad b_4 = a_3 = -c_3 = \mu \tag{6.13}$$

where ν and μ are some numbers. The relations (6.12) and (6.13) are limitations on the coefficients of (6.10). If the functions $u_j^{(2)}$ are chosen in agreement with the choice of $z_i^{(2)}$ in (6.10), then in all noncritical equations, all the terms of second order depending only on ξ and η are eliminated after Liapunov's transformation. We shall introduce Expression (6.13) into (6.10) and we shall substitute the obtained functions into (6.4)

$$\frac{d\xi}{dt} = -\eta + \frac{\eta^3}{6} - 2k(\nu\xi^2 + b_1\xi\eta - \nu\eta^2)(\mu\xi^2 + b_3\xi\eta - \mu\eta^2) + Y^{(5)}, \quad \frac{d\eta}{dt} = \xi \tag{6.14}$$

We take the Liapunov function

$$V = \xi^2 + \eta^2 - {}^1/_{12}\eta^4 \tag{6.15}$$

We compute the derivative of (6.15) on the basis of (6.14) and require the fulfillment of the expression

$$\left(\frac{dV}{dt}\right)^{(5)} = -4k\xi\,(\nu\xi^2 + b_1\xi\eta - \nu\eta^2)\,(\mu\xi^2 + b_3\xi\eta - \mu\eta^2) =$$
$$= -\psi_5(\xi,\eta) = -\gamma_1\xi^5\xi_* - \gamma_2\xi^4\eta\eta_* - \gamma_3\xi^3\eta^2\xi_* - \gamma_4\xi^2\eta^3\eta_* - \gamma_5\xi\eta^4\xi_* - \gamma_6\eta^5\eta_* \quad (6.16)$$

Since we must have $\gamma_i \geqslant 0$ $(i = 1,\ldots,6)$ in (6.16) and since ν and μ are numbers, then, in order to satisfy (6.16) it is necessary to take $\nu = \mu = \gamma_1 = \gamma_2 = \gamma_4 = \gamma_5 = \gamma_6 = 0$, and to satisfy the condition

$$-4kb_1b_3\xi^3\eta^2 = -\gamma_3\xi^3\eta^2\xi_* \quad (6.17)$$

We take $b_1 = \xi_*$, $b_3 = 1$; then $\gamma_3 = 4k > 0$. Substituting the found values for a_1, b_1, c_1 into (6.10) we get from Equation (2.6) some relations for the choice of $u_j^{(2)}$ $(j = 1, 2)$. The remaining values a_2, c_2, a_4, c_4 must be chosen to satisfy the conditions (6.12) in which $b_1 = \xi_*$, $b_3 = 1$. We shall take for instance $a_2 = -{}^1/_2\xi_*$, $a_4 = -{}^1/_2$, $c_2 = c_4 = 0$; then we obtain

$$u_1^{(2)} = {}^1/_2\xi^2\xi_* + 2\xi\eta\xi_* - \eta^2\xi_*, \qquad u_2^{(2)} = {}^1/_2\xi^2 + 2\xi\eta - \eta^2 \quad (6.18)$$

Taking, according to (4.18), the Liapunov function

$$V^* = V + a\xi\eta^5 - b\xi^5\eta \quad (6.19)$$

we find on the basis of (6.14) that it has a negative definite derivative for $a = b = 1$. If we change the terms of fifth order in (6.14), we may always choose on the basis of (4.21) quantities a and b in (6.19) such, that dV^*/dt be negative definite. Consequently, when a control is chosen in agreement with (6.5), (6.18) the simplified system (6.14) becomes asymptotically stable independently of terms of order higher than the fourth. The Liapunov transformation is determined by the continuous functions $z_1 = \xi\eta\xi_*$, $z_2 = -{}^1/_2\xi^2\xi_*$, $z_3 = \xi\eta$, $z_4 = -{}^1/_2\xi^2$. Carrying out transformation, we get in the noncritical equations terms of the third order $2k\xi^2\eta\xi_*$, $X_1^{(3)*}$ in the equations for x_1, x_3 which include the controls u_1, u_2. The asymptotic stability of the simplified system is determined by terms of order not higher than $N = 4$. For our example, $q = r = 2$, and according to (2.14) $p \geqslant {}^1/_2(4 + 1 - 2 + 4) = 3.5$, i.e. $p = 4$. Expressing the terms of third order in Equation (2.6), it can be seen that for $u_1^{(3)} = c = u_2^{(3)} \equiv 0$, the terms of third order in the functions z_i (2.7) are discontinuous; consequently the condition (2.15) cannot be applied and it is necessary to suppress the factors $2k\xi^2\eta\xi_*$, $X_1^{(3)*}$ by means of the controls u_1, u_2. Thus the controller

$$u_1 = -2x_1 - x_2 + {}^1/_2\xi^2\xi_* + 2\xi\eta\xi_* - \eta^2\xi_* - X_1^{(3)*}(\xi,\eta)$$
$$u_2 = -2x_3 + {}^1/_2\xi^2 + 2\xi\eta - \eta^2 - 2k\xi^2\eta\xi_*$$

stabilizes the pendulum on the basis of the complete system of equations.

2. Stabilization by the controller (2.16). In (6.2), (6.3) and (6.4) we shall discard the terms of order higher than the second and we shall substitute the functions

$$z_1 = \frac{\xi\eta}{\sqrt{\alpha\xi^2 + \beta\eta^2}}, \qquad z_{3} = \eta \quad (6.20)$$

in place of x_1 and x_3 in the equations for the critical variables (6.4).

On the basis of the obtained equations the derivative of the Liapunov function $V = \xi^2 + \eta^2$ is always negative

$$\left(\frac{dV}{dt}\right)^{(3)} = -4k\xi z_1 z_3 = -\frac{4k\xi^2\eta^2}{\sqrt{\alpha\xi^2 + \beta\eta^2}} \leqslant 0 \quad (6.21)$$

We take the functions

$$z_2 = \frac{a\xi^2 + b\xi\eta + c\eta^2}{\sqrt{\alpha\xi^2 + \beta\eta^2}}, \qquad z_4 = p\xi + q\eta \quad (6.22)$$

We select the coefficients $a, b, c, p, q, \alpha, \beta$ in (6.22) and the controls u_1, u_2, of the form (2.16) so that after the Liapunov transformation (2.10), the noncritical equations (6.2), (6.3) do not contain terms of first order depending only on ξ, η. This can be obtained, writing for instance,

$\alpha = \beta,\ b = c = q = 0,\ p = -1,\ a = -1/_2$ and taking the controls

$$u_1 = -2x_1 - x_2 + \frac{1/_2\,\xi^2 + 2\xi\eta - \eta^2}{\sqrt{\alpha}\,\sqrt{\xi^2 + \eta^2}}, \qquad u_2 = -2x_3 + 2\eta \tag{6.23}$$

Choosing in agreement with (4.18) the Liapunov function $V^* = V + \nu\xi\eta^3 - \mu\xi^3\eta$, it can be verified that the motion $\xi = \eta = 0$ of the simplified system obtained by using the control (6.23) in (6.2), (6.3), (6.4), is asymptotically stable, independently from the terms of order higher than the second. As far as the Liapunov transformation (2.10) is continuous in the present case, we may replace the conditions (2.14) by the condition (2.15), according to which for $N = 2$ we have $p \geqslant 1/_2\,(2 + 1 - 2 + 2) = 1.5$, i.e. $p = 2$. Consequently, the terms of order higher than the first in noncritical equation do not perturbe the asymptotic stability and the controller (6.23) stabilizes the pendulum on the basis of the total system of equations.

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